The rotation matrix may be read from the last equation as

$$R_{ij} = (Q_0^2 - Q_k Q_k)\delta_{ij} + 2Q_i Q_j - 2\varepsilon_{ijk} Q_0 Q_k \qquad \text{(A21)}$$

or more explicitly as

$$\boldsymbol{R} = \begin{Vmatrix} Q_0^2 + Q_x^2 - Q_y^2 - Q_z^2 & 2Q_xQ_y - 2Q_zQ_0 & 2Q_xQ_z + 2Q_yQ_0 \\ 2Q_yQ_x + 2Q_zQ_0 & Q_0^2 - Q_x^2 + Q_y^2 - Q_z^2 & 2Q_yQ_z - 2Q_xQ_0 \\ 2Q_zQ_x - 2Q_yQ_0 & 2Q_zQ_y + 2Q_xQ_0 & Q_0^2 - Q_x^2 - Q_y^2 + Q_z^2 \end{Vmatrix} \qquad \text{(A22)}$$

Note that the quaternion representing the null rotation, obtained by substituting $\alpha = 0$ in Eq. (A15), is the number 1. But a rotation by 2π gives rise to $Q = -1$. However, the rotation induced by Q and by $-Q$ are the same. The sign cancels in Eqs. (A20), (A21), and (A22), which are all quadratic in the components of Q.

In summary, unit quaternions describe rotations in three dimensions. The quaternion corresponding to a given rotation is unique up to a sign. Once the quaternion is known, computation of the effect of the rotation requires no trigonometry. Rotations may be combined by quaternion multiplication. Quaternion history may be derived from the history of angular velocity using the differential Eqs. (A18) and (A19).

Quaternions describe the three degrees-of-freedom of rotation in three dimensions by four parameters (the quaternion components) subject to one constraint:

$$|Q|^2 = (QQ^*) = Q_0^2 + \boldsymbol{Q}\cdot\boldsymbol{Q} = 1 \qquad \text{(A23)}$$

The constraint is self-preserving. The differential Eqs. (A18) and (A19) assure that the constraint, if satisfied initially, will be satisfied for all time.

When computers are used to integrate the differential equations, truncation errors can cause violation of the constraints. If unchecked, such violations would statistically grow as time increases. For this reason it is necessary to reinforce the constraints at intervals, normally at every integration step.

The requirement that a quaternion obtained from integrating Eqs. (A18) and (A19) be a unit quaternion may be reinforced by renormalization. That is, each of the four components is multiplied by

$$|Q|^{-1} = \{Q_0^2 + \boldsymbol{Q}^2\}^{-1/2} \qquad \text{(A24)}$$

However, $|Q|$ is virtually equal to 1, any difference being merely the effect of truncation error for one integration step. For this reason the full computational burden of Eq. (A24) (including a square root and a division) is not necessary. Instead, one may expand in the small difference and retain terms only up to first order:

$$|Q|^{-1} = \{1 - (1 - Q_0^2 - \boldsymbol{Q}^2)\}^{-1/2}$$
$$\approx 1 + \tfrac{1}{2}(1 - Q_0^2 - \boldsymbol{Q}^2) = \tfrac{3}{2} - \tfrac{1}{2}(Q_0^2 + \boldsymbol{Q}^2) \qquad \text{(A25)}$$

The last factor, which is vitually unity, suffices to enforce the constraint for all times. The burden of computing it involves only multiplications and additions.

Appendix B: Special Rotation Vectors

Special rotation vectors are three-dimensional vectors of length unity or less:

$$|\boldsymbol{R}| \leq 1 \qquad \text{(B1)}$$

Each SRV defines a unique rotation as follows: The axis of the rotation is in the direction of $\boldsymbol{R}$; the angle of the rotation α is determined by

$$0 \leq \alpha \leq \pi \qquad \text{(B2)}$$

$$\sin(\tfrac{1}{2}\alpha) = |\boldsymbol{R}| \qquad \text{(B3)}$$

The rotation is taken in the positive screw sense around $\boldsymbol{R}$. Only the range in Eq. (B2) need be included, since rotations described by an SRV in the opposite direction cover the complementary range. The origin corresponds to no rotation (reference orientation). The points on the surface of the sphere describe rotations by π. Opposite points on the surface of the sphere, corresponding to rotations by 180 deg in opposite directions, describe the same orientation. Each pair of opposite surface points defined a single SRV. However, in practice, any one can be used.

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Total Least Squares Estimation of Aerodynamic Model Parameters from Flight Data

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Introduction

EQUATION-ERROR techniques for the identification of aircraft models from flight test data were introduced in the 1950s. Greenberg[1] applies these techniques on the problem of identifying parameters in the aircraft's differential equations-of-motion, later Gerlach[2] shows applications to linear aerodynamic equations. Both applications are based on the method of "least squares" introduced by Gauss[3] in 1809.

One way of assessing the accuracy of the estimated parameters is through consulting the least squares generated variance matrix. Another way is through repeating the experiment and observing the scatter in the estimated parameters. Both assessments are often found to disagree. The calculated variance matrix may predict a high accuracy, while the scatter in the estimated parameters suggests the opposite.

This Note shows an example where this controversy was caused by the incapability of the least squares method to

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account correctly for multiple sources of measurement errors encountered in a flight test instrumentation system. This problem is then solved through the application of the "total least squares" method.[4]

System Identification Theory

Least Squares Estimation (LS)

Let

$$A\theta \approx y \tag{1}$$

be a linear model, where y is the vector of observations to be explained, A is a $(m \times n)$ matrix of explanatory variables, and θ is the unknown parameter vector to be estimated. Gauss[3] chose the most probable values of the parameters as the estimation objective. With the assumption of a normal distribution of the errors Δy on the variable-to-be-explained y, the original estimation objective translates into minimizing the following quadratic cost functional:

$$J_{\mathrm{LS}}(\theta) = [y - A\theta]^T W^{-1}[y - A\theta] \tag{2}$$

where, W denotes the error variance matrix $E\{\Delta y \Delta y^T\}$. Zeroing the gradient of the cost functional with respect to the unknown parameters θ, yields the well-known (weighted) least squares estimation algorithm

$$\hat{\theta}_{\mathrm{LS}} = (A^T W^{-1} A)^{-1} A^T W^{-1} y \tag{3}$$

$$C0v\ \hat{\theta}_{\mathrm{LS}} = (A^T W^{-1} A)^{-1} \tag{4}$$

Total Least Squares Estimation (TLS)

In many applications, the matrix of explanatory variables A may contain variables which are subject to measurement errors ΔA. Least squares does not explicitly account for these additional error sources and, consequently, provides incorrect information on the parameter estimates as well as the calculated variance matrix. To overcome this problem, the total least squares method was introduced quite recently in the numerical analysis field by Golub and Van Loan.[4]

The basic principle of TLS is to take a unified approach to the data A and output y by restating the problem of Eq. 1 as

$$[A|y][\theta^T|-1]^T \approx 0 \tag{5}$$

Due to the measurement errors, the compound data matrix $[A|y]$ will be of full rank and there is no nonzero solution vector $[\theta^T|-1]^T$, therefore, Eq. (5) is incompatible. In order to force a solution, the rank of the measured data $[A|y]$ must be reduced with an estimate of the data errors $[\Delta A|\Delta y]$

$$([A|y] - [\Delta\hat{A}|\Delta\hat{y}])[\theta^T|-1]^T = 0 \tag{6}$$

subject to the constraint of minimal approximation effort

$$\|[\Delta\hat{A}|\Delta\hat{y}]C^{-1}\|_F^2 \quad \text{is minimal} \tag{7}$$

where, C is a square root of the covariance matrix of the row vectors of the data error matrix $[\Delta A|\Delta y]$. The solution to this rank reduction problem is found in terms of the singular value decomposition of the matrix $[A|y]C^{-1}$

$$[A|y]C^{-1} = U\Sigma V^T, \qquad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) \tag{8}$$

where $\sigma_1, \ldots, \sigma_{n+1}$ are the singular values, U and V are the left and right singular matrices of $[A|y]C^{-1}$. The smallest singular value σ_{n+1} is the Frobenius norm distance of $[A|y]C^{-1}$ to the nearest rank deficient matrix $[A - \Delta\hat{A}|y - \Delta\hat{y}]C^{-1}$. Hence, the estimated data correction and the corrected data are given by

$$[\Delta\hat{A}|\Delta\hat{y}]C^{-1} = U\Delta\hat{\Sigma}V^T, \qquad \Delta\hat{\Sigma} = \mathrm{diag}(0, \ldots, 0, \sigma_{n+1})$$

$$[\hat{A}|\hat{y}]C^{-1} = U\hat{\Sigma}V^T, \qquad \hat{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, 0) \tag{9}$$

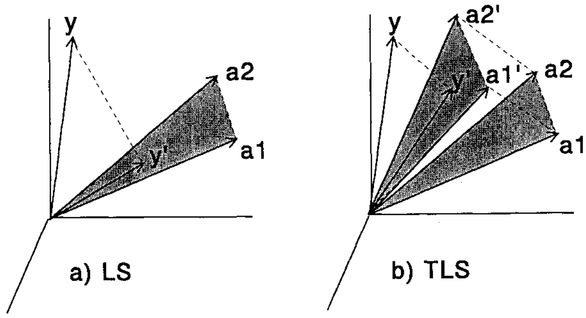

Fig. 1 Geometrical interpretation of a) LS and b) TLS.

Once the data is corrected with an estimate of the errors, the transformed solution vector $C[\theta^T|-1]$ is found as the kernel of matrix $[\hat{A}|\hat{y}]C^{-1}$, which equals the last column vector v_{n+1} of the right singular matrix V

$$C[\hat{\theta}_{\mathrm{TLS}}^T|-1]^T = \lambda v_{n+1} \tag{10}$$

Eq. 10 is then solved for $[\hat{\theta}_{\mathrm{TLS}}^T|-1]^T$, whereby λ is chosen such that the last element equals -1.

The covariance matrix of the total least squares estimate was calculated by Gleser[5] and later simplified by Van Huffel[6]

$$C0v\ \hat{\theta}_{\mathrm{TLS}} = [\hat{\theta}_{\mathrm{TLS}}^T|-1]C^T C \begin{bmatrix} \hat{\theta}_{\mathrm{TLS}} \\ -1 \end{bmatrix} [A^T A - E\{\Delta A^T \Delta A\}]^{-1} \tag{11}$$

LS vs TLS, a Geometrical Interpretation

Van Huffel[6] introduces a geometrical comparison of the least squares and the total least squares principles seen from the column space of the compound data matrix $[A|y]$.

Figure 1a shows the classical principle of least squares estimation which is to apply an orthogonal projection of the vector-in-error y on the column space of matrix A.

In the total least squares principle (Fig. 1b), it is realized that the observed vector y as well as the explanatory variables $A = [a_1, a_2, \ldots]$ are in error and, therefore, should all "bend" towards each other to form a new, compatible, set of equations.

Application to Flight Test Data Analysis

Aerodynamic Model

As a concrete example, the LS/TLS theory will be applied on the identification of the following aerodynamic force equation of the Swearingen Metro II

$$C_Z = \frac{A_Z m}{\frac{1}{2}\rho V^2 S} = C_{Z0} + C_{Z_\alpha}\alpha + C_{Z_q}\frac{q\bar{c}}{V} + C_{Z_{\delta_e}}\delta_e \tag{12}$$

During a flight test program, carried out in cooperation with the National Aerospace Laboratory in The Netherlands, measurements were made of the specific force A_Z, airspeed V, angle of attack α, pitch rate q, and elevator deflection δ_e. The purpose of the estimation process is to find the most probable values of the parameters C_{Z0}, C_{Z_α}, C_{Z_q}, and $C_{Z_{\delta_e}}$.

Measurement Error Variances

Total least squares requires knowledge of the error variances of all measured variables in Eq. (12). The instrumentation department of the National Aerospace Laboratory supplied us with the transducer accuracies shown in Table 1.

Care is required in using angle-of-attack and airspeed data. These data stem from measuring pressure and flow direction in the direct vicinity of the aircraft, which may give rise to systematic errors. TLS accounts for random errors only and is unable to handle any systematic errors. Therefore, airspeed and angle-of-attack data were calibrated carefully.

Table 1 Standard deviation of measurement errors

Specific force	A_z	0.012 m/s²
Pitch rate	q	0.017 deg/s
Airspeed	V	0.150 m/s
Angle of attack	α	0.030 deg
Elevator deflection	δ_e	0.014 deg

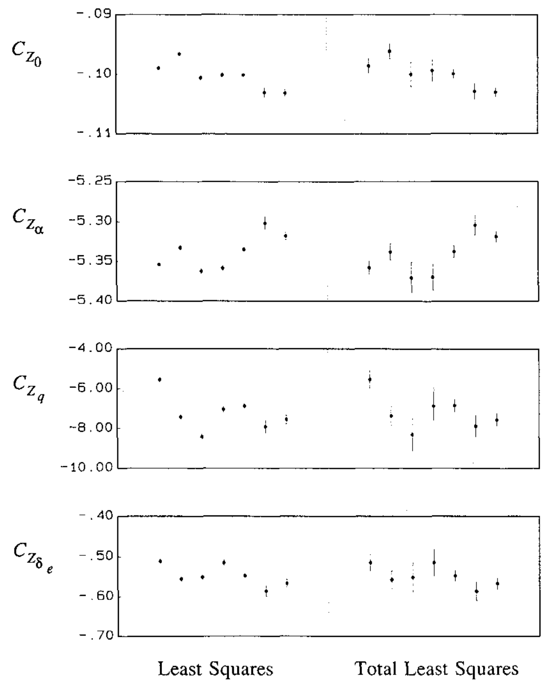

Fig. 2 LS/TLS estimation results C_Z equation. [Dot = estimated parameter value, Eqs. (3) and (10), and bar = predicted standard deviation, Eqs. (4) and (11).]

Parameter Identification Results

Data from 7 elevator doublets and 3211 flight test maneuvers were collected and analyzed with least squares and total least squares. The results are shown in Fig. 2.

The first column of Fig. 2 shows the least squares estimation results. The theoretical standard deviations were calculated according to Eq. (4) and seem to predict a high accuracy of the LS parameter estimates. However, it falls short in explaining the true scatter seen between the parameter estimates from the seven different maneuvers.

The second column of Fig. 2 shows the total least squares estimation results. The TLS parameter estimates do not differ significantly from the LS results, which is due to the high signal-to-noise ratio obtained with flight test instrumentation systems. Yet, Fig. 2 shows how the theoretical TLS standard deviations, calculated according to Eq. (11), differ significantly from the LS results. TLS accounts for all measurement error sources correctly (Table 1), while LS ignores errors on the explanatory variables and accounts for errors on C_Z only. This results in an incorrect overconfidence in the LS results. Notice that the TLS standard deviations agree better with the observed scatter between the results of different maneuvers.

Conclusions

TLS estimation was introduced in the field of aircraft parameter identification. Its merit is found in accurately accounting for multiple sources of (random) errors in a flight test instrumentation system. Unlike LS estimation, TLS was shown to give realistic predictions of the standard deviations of its estimates.

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Effect of Viscous Drag on Optimum Spanwise Lift Distribution

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Introduction

TRADITIONALLY, Prandtl's[1] lifting line theory has been used in classical aerodynamics to show that the optimum spanwise lift distribution is an elliptic one. This theory is based on the assumption that the entire lift is concentrated on a single line spanning the wing. The wake is assumed to be flat, and the local flow over the wing is taken to be chordwise only. Under these assumptions, elliptic lift distribution is shown to minimize the induced drag.

Numerical techniques such as vortex lattice methods have offered new approaches to optimization. These methods allow the user to manipulate the chordwise as well as the spanwise distribution of lift for minimum induced drag. Examples can be found in Refs. 2 and 3. In these methods, the induced drag due to lift is modeled in a variational form with lift as the constraint and the induced drag as the quantity to be extremeized. Another approach, based on a modified version of Multihopp's lifting surface theory can also be found in Ref. 4.

The general logic in all these cases has been that the viscous and induced drags can be optimized and then added directly. One of the flaws of this approach is that the spanwise lift distribution is based on circulation that is generated within the boundary layer. The same circulation (and therefore, the boundary layer) is also responsible for viscous drag. Therefore, the spanwise lift distribution that minimizes the induced drag may not necessarily minimize the viscous drag, and therefore, the total drag. Subsequently, it is the intention of this note to show how Prandtl's method can be modified to account for viscous drag. It will be demonstrated that the inclusion of the viscous drag renders the elliptic lift distribution nonoptimal. It will also be shown that elliptic lift distribution, although nonoptimal, is quite close to the optimum case.

Method of Analysis

The development presented here very closely follows those of standard texts in aerodynamic theory, such as Refs. 5 and

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